



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 43

Option Pricing – Binomial Model – 1

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OPTION PRICING



BASIC OPTION PRICING MODELS

- THE BINOMIAL MODEL
- THE BLACK SCHOLES MODEL



THE FUNDAMENTAL ASSUMPTION

- We can create a risk free asset through a combination of a long/short position in a derivative and an appropriate position in the underlying asset i.e. that it is possible to annul the volatility in the price of the derivative by a counter position in the underlying asset and vice versa.
- This is possible because the source of randomness in both price processes is modeled identically.



THE SINGLE PERIOD BINOMIAL PRICING MODEL

- **WE MODEL THE STOCK PRICE AS SINGLE STEP RANDOM WALK**
- Consider a stock priced at S_0 at $t=0$.
- Let, at the end of one period of length T i.e. at $t=T$, the stock price take one of two possible values viz. $S_u=S_0u$ or $S_d=S_0d$.
- **The stock can take no other value except S_u or S_d at $t=T$.**
- **There will be no change in stock value in the interval $(0,T)$ since this is a single period model.** Only at the end of this single period i.e. at $t=T$, the stock value will jump from S_0 to either S_u or S_d .



THE RISKLESS HEDGE

- We construct a portfolio V at $t=0$ consisting of:
- (i) one unit of derivative, costing f_0 at $t=0$ and
- (ii) Δ units of stock costing S_0 .
- The value of this portfolio at $t=0$ is $V_0 = f_0 + \Delta S_0$.



- The value of this portfolio at $t=T$ will be:
- $V_T^u = f_T^u + \Delta S_u$ if the stock price goes up and
- $V_T^d = f_T^d + \Delta S_d$ if the stock price goes down
- where f_T^u and f_T^d are the payoffs from the derivative at $t=T$ if the stock price goes up and down respectively at $t=T$.

- Now, if the portfolio V is to be riskless, its value at $t=T$ must be independent of the up and down movement of the stock price.
- This gives:
- $V_T^u = f_T^u + \Delta S_u = f_T^d + \Delta S_d = V_T^d$
- so that $\Delta = (-) (f_T^u - f_T^d) / (S_u - S_d) = (-) \delta f / \delta S$.
- The negative sign indicates that the position in the underlying will be opposite to that of the derivative.
- Also since portfolio V is riskless, it will generate the riskfree rate of return.

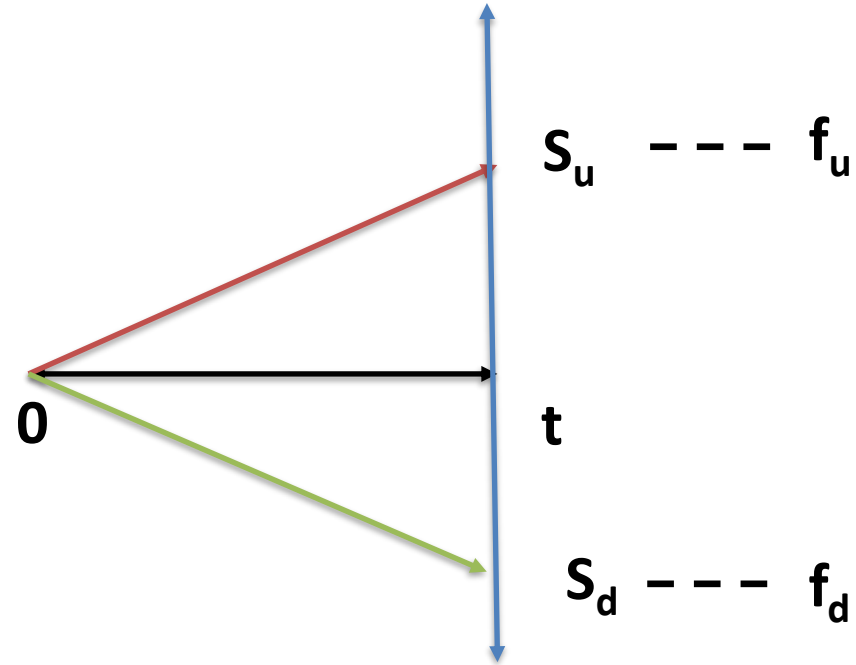
$$V_0 = f_0 + \Delta S_0 \text{ or } f_0 = V_0 - \Delta S_0 \quad (1)$$

$$V_T = f_T^u + \Delta S_u = f_T^d + \Delta S_d \quad (2)$$

$$V_T = V_0 e^{rT} \text{ or } V_0 = V_T e^{-rT} \quad (3)$$

$$\text{From (2) } \Delta = - \frac{(f_T^u - f_T^d)}{S_u - S_d} \quad (4)$$

$$\begin{aligned} \text{giving } f_0 &= V_0 - \Delta S_0 = V_T e^{-rT} - \Delta S_0 \\ &= (f_T^u + \Delta S_u) e^{-rT} - \Delta S_0. \end{aligned}$$



$$f_0 = V_0 - \Delta S_0 = V_T e^{-rT} - \Delta S_0 = (f_T^u + \Delta S_u) e^{-rT} - \Delta S_0$$

Substituting $\Delta = \left[(-) \frac{(f_T^u - f_T^d)}{S_u - S_d} \right]$, we get

$$= e^{-rT} \frac{f_T^d S_u - f_T^u S_d}{S_u - S_d} + \frac{(f_T^u - f_T^d)}{S_u - S_d} S_0 = e^{-rT} \left[\frac{f_T^d S_u - f_T^u S_d}{S_u - S_d} + \frac{(f_T^u - f_T^d)}{S_u - S_d} S_0 e^{rT} \right]$$

$$\begin{aligned}
\mathbf{f}_0 &= \mathbf{e}^{-rT} \left[\frac{\mathbf{f}_T^d S_u - \mathbf{f}_T^u S_d}{S_u - S_d} + \frac{(\mathbf{f}_T^u - \mathbf{f}_T^d)}{S_u - S_d} S_0 \mathbf{e}^{rT} \right] \\
&= \mathbf{e}^{-rT} \left[\mathbf{f}_T^d \left(\frac{S_u - S_0 \mathbf{e}^{rT}}{S_u - S_d} \right) + \mathbf{f}_T^u \left(\frac{S_0 \mathbf{e}^{rT} - S_d}{S_u - S_d} \right) \right] \\
&= \mathbf{e}^{-rT} \left[\mathbf{q}_d \mathbf{f}_T^d + \mathbf{q}_u \mathbf{f}_T^u \right] = \mathbf{e}^{-rT} \mathbf{E}_Q \left[\mathbf{f}(S_T) \right] \text{ where} \\
\mathbf{q}_d &= \left(\frac{S_u - S_0 \mathbf{e}^{rT}}{S_u - S_d} \right) \text{ and } \mathbf{q}_u = \left(\frac{S_0 \mathbf{e}^{rT} - S_d}{S_u - S_d} \right)
\end{aligned}$$

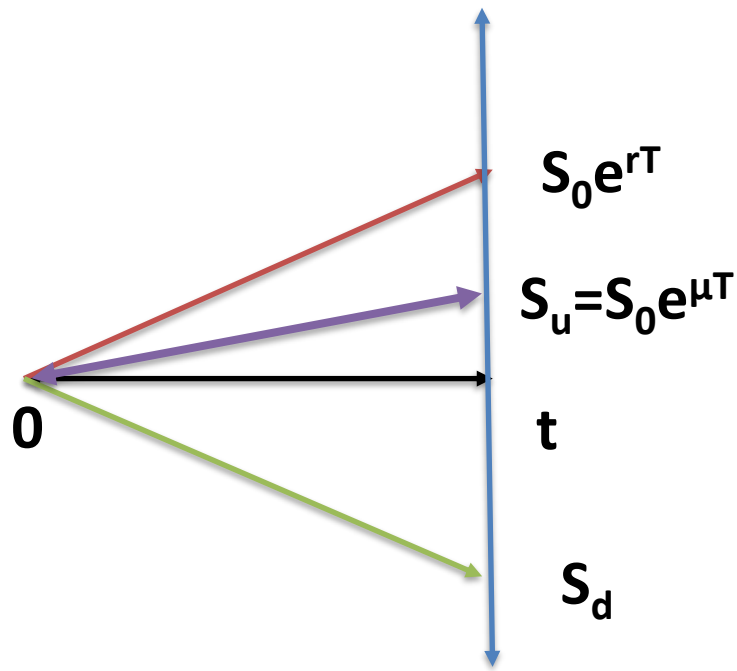
SYNTHETIC (Q) PROBABILITIES

We have $q_d + q_u = \left(\frac{S_u - S_0 e^{rT}}{S_u - S_d} \right) + \left(\frac{S_0 e^{rT} - S_d}{S_u - S_d} \right) = 1$

Also $0 \leq \left(\frac{S_u - S_0 e^{rT}}{S_u - S_d} \right) \leq 1$; Why?

$0 \leq \left(\frac{S_0 e^{rT} - S_d}{S_u - S_d} \right) \leq 1$; Why?





Let $\left(\frac{S_0 e^{rT} - S_d}{S_u - S_d} \right) > 1$ so that

$$S_0 e^{rT} > S_u = S_0 e^{\mu T}.$$

Thus, we can get riskfree return r greater than maximum risky return μ .

By no arbitrage, this is not possible.